§9 discrete valuation rings and Dedekind domain

Last chapter: Noetherian rings of $\underline{\dim 0}$. $\forall \mathfrak{p} = \text{maximal}$

Question: Noetherian $\underline{\text{integral domains of dim 1}}$. ?

$\mathfrak{p} = \text{prime} \Rightarrow \mathfrak{p} = 0$ or $\mathfrak{p} = \text{maximal}$

Prop 9.1 (unique factorization thm) Let A be a Noetherian domain of dim 1. every non-zero ideal can be $\underline{\underline{\text{uniquely}}}$ expressed as a $\underline{\underline{\text{product}}}$ of primary ideals whose radicals are distinct.

Pf: $0 \neq \mathfrak{a} \lhd A$

Noetherian $\overset{7.13}{\Rightarrow}$ $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ minimal primary decomp.

(WTS: $\mathfrak{a} = \prod_{i=1}^{n} \mathfrak{q}_i$)

$\left.\begin{array}{l} \mathfrak{p}_i := \sqrt{\mathfrak{q}_i} \neq 0 \text{ prime} \\ \dim A = 1 \end{array}\right\} \Rightarrow \mathfrak{p}_i = \max.$

$\Rightarrow \mathfrak{p}_i + \mathfrak{p}_j = A$

$\overset{1.16}{\Rightarrow} \mathfrak{q}_i + \mathfrak{q}_j = A$

$\overset{1.10}{\Rightarrow} \prod_i \mathfrak{q}_i = \bigcap_i \mathfrak{q}_i = \mathfrak{a}.$

Conversely. $\mathfrak{a} = \prod_i \mathfrak{q}_i \qquad \sqrt{\mathfrak{q}_i} \neq \sqrt{\mathfrak{q}_j}$

$\Rightarrow \mathfrak{a} = \bigcap_i \mathfrak{q}_i$ minimal primary decomp.

$\Rightarrow \forall i$, $\mathfrak{q}_i$ is isolated

$\overset{4.1}{\Rightarrow}$ uniqueness $\square$

Fact: assume every primary ideal is a prime power, then

prop 8.1 $\Rightarrow \quad \mathfrak{a} = \prod_i \mathfrak{p}_i^{n_i}$

prime

§9.2 discrete valuation rings.

K = field.

A discrete valuation on K is a surj. mapping

$$v : K^* \twoheadrightarrow \mathbb{Z} \qquad (K^* = K - \{0\})$$

surjective

s.t.

1) $v(xy) = v(x) + v(y)$

2) $v(x+y) \geq \min(v(x), v(y))$

valuation ring of v:

$$\{ x \in K^* \mid v(x) \geq 0 \} \cup \{0\}$$

$$(v : K \to \mathbb{Z} \cup \{+\infty\}, \quad v(0) = +\infty)$$

Example: 1) $K = \mathbb{Q}$, $\quad p \nmid mn \Rightarrow v_p\left(p^a \cdot \frac{m}{n}\right) := a$

$$\mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \mid (p,n) = 1 \right\}$$

2) $K = k(x)$, $\quad f$ = irr. $\quad f \nmid gh \Rightarrow v_f\left(f^a \frac{g}{h}\right) = a$

$$k[x]_{(f)} = \left\{ \frac{g}{h} \mid (f,h) = 1 \right\}$$

Fact: discrete valuation rings.

$(5.18) \Rightarrow$ local with maximal ideal

$$\mathfrak{m} := \{ x \in K \mid v(x) > 0 \}$$

Prop 9.2. $(A, \mathfrak{m}, k)$ = Noether + local domain + dim 1. TFAE:

$\mathfrak{m}$: max. ideal, k: residue field.

i) $A = \mathrm{DVR}$

ii) A = integrally closed

iii) $\mathfrak{m}$ = principal ideal

iv) $\dim_k (\mathfrak{m}/\mathfrak{m}^2) = 1$

v) $\forall\, \mathfrak{a} \underset{\neq 0}{\triangleleft} A \Rightarrow \mathfrak{a} = \mathfrak{m}^r$ for some r

vi) $\exists\, x \in A$ s.t. $\forall\, \mathfrak{a} \underset{\neq 0}{\triangleleft} A$ $\exists\, r$ s.t. $\mathfrak{a} = (x^r)$.

Two remarks:

$(7.16) \Rightarrow$ (A) $\quad \mathfrak{a} \triangleleft A,\ \mathfrak{a} \neq 0, A \Rightarrow \begin{cases} \mathfrak{a} = \mathfrak{m}\text{-primary} \\ \mathfrak{a} \supseteq \mathfrak{m}^n \quad n \gg 0. \end{cases}$

$(8.6) \Rightarrow$ (B). $\quad \mathfrak{m}^n \neq \mathfrak{m}^{n+1} \quad \forall\, n \geq 0.$

Pf: i) $\overset{5.18}{\Rightarrow}$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv) $\Rightarrow$ v) $\Rightarrow$ vi) $\Rightarrow$ i)

i) $\overset{5.18}{\Rightarrow}$ ii) $\quad \alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0 \Rightarrow \alpha = -a_0(\alpha^{-1})^{n-1} - \cdots - a_{n-1}$

ii) $\Rightarrow$ iii) $\forall\, a \in m - \{0\}$.

$(A) \Rightarrow \exists n$ s.t. $m^n \subseteq (a)$ & $m^{n-1} \nsubseteq (a)$

$\forall\, b \in m^{n-1} - (a) \neq \phi$

$x := \frac{a}{b} \in K = \mathrm{Frac}\, A$

$b \notin (a) \Rightarrow x^{-1} \notin A$

$\overset{5.1}{\Rightarrow} x^{-1} m \nsubseteq m$ $\left(\begin{array}{l} \text{or } x^{-1}m \subseteq m \\ \Rightarrow m = \text{faithful } A[x^{-1}]\text{-mod.} \\ \quad \& \text{ f.g. } A\text{-mod} \end{array}\right)$

Construction of $x \Rightarrow x^{-1}m \subseteq A$ $\left(\begin{array}{l} \forall\, y \in m \\ y \cdot a x^{-1} = yb \in m^n \subset (a) \\ \Rightarrow y x^{-1} \in A \end{array}\right)$

$\Rightarrow x^{-1}m = A$

$\Rightarrow m = (x)$.

iii) $\Rightarrow$ iv) $\left.\begin{array}{l} (2.8) \Rightarrow \dim_k (m/m^2) \leq 1 \\ (B) \Rightarrow m \neq m^2 \end{array}\right\} \Rightarrow V$

iv) $\Rightarrow$ v) $\quad \mathfrak{a} \neq (0), A \Rightarrow \exists\, n$ s.t. $\mathfrak{m}^n \subseteq \mathfrak{a}$.

$$\overset{(8.8)}{\Rightarrow} \mathfrak{a}/\mathfrak{m}^{n+1} \lhd A/\mathfrak{m}^{n+1} \text{ is principle.}$$

$$\Rightarrow \mathfrak{a} = (x) + \mathfrak{m}^{n+1}$$

$$= (x) + \mathfrak{m}\mathfrak{a}$$

$$\Rightarrow \mathfrak{a} = (x)$$

v) $\Rightarrow$ vi) $\quad (B) \Rightarrow \mathfrak{m} \neq \mathfrak{m}^2 \Rightarrow \exists\, x \in \mathfrak{m} - \mathfrak{m}^2$

$$\overset{\text{hypothesis}}{\Longrightarrow} \exists r, \text{ s.t. } (x) = \mathfrak{m}^r$$

$$\overset{x \notin \mathfrak{m}^2}{\Longrightarrow} r = 1$$

$$\Rightarrow \mathfrak{m}^k = (x^k) \quad \forall\, k$$

vi) $\Rightarrow$ i). $\quad \mathfrak{m} = (x)$. $\quad (x^k) \neq (x^{k+1}) \ (\Leftarrow (B))$

$\forall a \in A - \{0\} \Rightarrow \exists!\, k$ s.t. $(a) = (x^k)$

$$\begin{cases} v(a) := k \\ v(ab^{-1}) = v(a) - v(b) \quad \text{on } K^* \end{cases}$$

- well-defined
- discrete
- A = the valuation ring of v. $\quad \square$

§9.3 Dedekind domains.

Thm 9.3 A = Noetherian domain & $\dim A = 1$. TFAE

1) A = integrally closed
2) $\forall \mathfrak{q} \triangleleft A$ primary $\Rightarrow \mathfrak{q} = \mathfrak{p}^m$ for some m. ($\mathfrak{p} = \sqrt{\mathfrak{q}}$)
3) $A_{\mathfrak{p}}$ = DVR $\quad \forall \mathfrak{p} \neq 0$

Pf: A = integrally closed $\overset{5.13}{\Longleftrightarrow}$ $A_{\mathfrak{p}}$ = integrally closed $\forall \mathfrak{p} \neq 0$.

$\Updownarrow$ 9.2

$A_{\mathfrak{p}}$ = DVR $\quad \forall \mathfrak{p} \neq 0$

$\Updownarrow$

$\forall$ $\mathfrak{p}$-primary is power of $\mathfrak{p}$ $\underset{\substack{4.8 \\ 3.11}}{\Longleftrightarrow}$ $\forall \mathfrak{b} \triangleleft A_{\mathfrak{p}}$, $\mathfrak{b} = \mathfrak{p}^m$

$$\begin{array}{ccc}
\{\mathfrak{p}\text{-primary}\} & \underset{4.8}{\overset{1:1}{\longleftrightarrow}} & \{S^{-1}\mathfrak{p}\text{-primary}\} \\
\downarrow & & \downarrow \\
\{\text{contracted}\} & \underset{3.11}{\overset{1:1}{\longleftrightarrow}} & \{\text{ideal of } S^{-1}A\}
\end{array}$$

Dedekind domain = ring satisfying conditions of (8.3).

<u>Cor</u> 8.4 (Unique factorization in Dedekind domain) A = Dedekind

$0 \neq \mathfrak{a} \triangleleft A \Rightarrow \mathfrak{a} = \mathfrak{p}_1^{\alpha_1} \mathfrak{p}_2^{\alpha_2} \cdots \mathfrak{p}_n^{\alpha_n}$

Pf (8.1) & (8.3). □

Example PID. ($\Rightarrow A_{\mathfrak{p}} = \text{PID} \Rightarrow A_{\mathfrak{p}} = \text{DVR} \Rightarrow A = \text{Dedekind}$)

Thm 8.5. · K = algebraic number field. i.e. $\mathbb{Q}[x]/(f)$ ← irr

· $\mathcal{O}_K := \{\alpha \in K \mid \alpha \text{ integral over } \mathbb{Z}\}$.

ring of integers in K

$\Rightarrow \mathcal{O}_K$ = Dedekind.

Pf: NTS: $\mathcal{O}_K$ = noetherian, dim 1, & integrally closed ✓

· char = 0 $\Rightarrow K$ = separable extension of $\mathbb{Q}$

$\overset{5.17}{\Rightarrow} \mathcal{O}_K$ = f.g. as a $\mathbb{Z}$-module

$\Rightarrow \mathcal{O}_K$ = Noetherian

· $\forall \underset{\neq 0}{\mathfrak{p}} \triangleleft \mathcal{O}_K \underset{\text{integral}}{\Rightarrow} \mathfrak{p} \cap \mathbb{Z} \neq 0 \Rightarrow \mathfrak{p} \cap \mathbb{Z} \triangleleft \mathbb{Z}$ maximal

$\Rightarrow \mathfrak{p}$ = maximal

$\Rightarrow \dim \mathcal{O}_K = 1$.

§ 8.4. fractional ideals

- A = integral domain, $K = \mathrm{Frac}\,A$,

$M \subseteq K$ is called a <u>fractional ideal</u> of A, if

- M = an A-submodule of K.
- $xM \subseteq A$ for some $x \in A \setminus \{0\}$.

· integral ideal (usual ideal)

Example (principal fractional ideal) $Au = \{au \mid a \in A\}$ $\forall u \in K$.

M = fractional ideal, recall

$$(A:M) := \{a \in K \mid aM \subset A\}$$ (different with that defined earlier)

Fact: f.g. A-submodule M of K is a fractional ideal

pf: $M = \sum_{i=1}^{n} A x_i \subseteq K$ $\quad x_i = \frac{y_i}{w_i}$, $y_i, w_i \in A$, $w_i \neq 0$

$\omega := \prod_{i=1}^{n} \omega_i$

$$\Rightarrow \omega M = \sum_{i=1}^{n} A\, y_i\, \omega_1 \cdots \omega_{i-1} \omega_{i+1} \cdots \omega_n \subseteq A.$$

Fact: Let $M, N \subseteq K$ be two submodules. $\underbrace{M \cdot N}_{\{\sum_{i=1}^r x_i y_i \mid x_i \in M,\, y_i \in N\}} = A$, Then

1) $M, N =$ fractional ideals
2) $M, N =$ f.g. A-mod.
3) $N = (A:M)$ & $M = (A:N)$

In this case, M, N are called Invertible ideal.

↳ $A = (1)$ identity

Pf: 1) clear

2) $1 = \sum_{i=1}^r x_i y_i \overset{\forall x \in M}{\Rightarrow} x = \sum (x y_i) \cdot x_i \in \sum_{i=1}^r A x_i$

$\Rightarrow M = \sum_{i=1}^r A x_i =$ f.g. A-module

3) $N \subseteq (A:M) = \underbrace{(A:M)MN}_{\subseteq A} \subseteq AN = N$

Example: non-zero principle $\Rightarrow$ invertible.

Prop 8.6 $M =$ fractional ideal. TFAE

i) $M =$ invertible

ii) $M =$ f.g. & $M_\mathfrak{p}$ invertible / $A_\mathfrak{p}$ $\quad \mathfrak{p} =$ prime

iii) $M =$ f.g. & $M_\mathfrak{m}$ invertible / $A_\mathfrak{m}$ $\quad \mathfrak{m} =$ max.

Pf: i) $\Rightarrow$ ii) $A_{\mathfrak{p}} = (M \cdot (A:M))_{\mathfrak{p}}$

$$\overset{3.11}{=} M_{\mathfrak{p}} \cdot (A:M)_{\mathfrak{p}}$$

$$\underset{3.15}{\overset{\text{f.g.}}{=}} M_{\mathfrak{p}} \cdot (A_{\mathfrak{p}} : M_{\mathfrak{p}})$$

$$\overset{\text{fact}}{\Rightarrow} \checkmark$$

ii) $\Rightarrow$ iii) $\checkmark$

iii) $\Rightarrow$ i) $\mathfrak{a} := M \cdot (A:M) \triangleleft A$

$$\Rightarrow \mathfrak{a}_{\mathfrak{m}} \overset{(3.11,\, 3.15)}{=} M_{\mathfrak{m}} \cdot (A_{\mathfrak{m}} : M_{\mathfrak{m}}) = A_{\mathfrak{m}} \quad \forall \mathfrak{m}$$

$$\Rightarrow \mathfrak{a} \not\subseteq \mathfrak{m} \quad \forall \mathfrak{m}$$

$$\Rightarrow \mathfrak{a} = A$$

<u>Prop 9.7</u> A = local domain.

A = DVR $\Leftrightarrow$ $\forall$ non-zero fractional ideal of A is invertible

Pf: $\Rightarrow$): $\mathfrak{m} = (x) \triangleleft A$

$\forall M \neq 0$ fractional ideal $\Rightarrow y \overset{\circ}{\overset{\#}{M}} \triangleleft A \Rightarrow yM = (x^r) \triangleleft A$

$$\Rightarrow M = (x^{r-s}), \quad s = v(y).$$

$\Leftarrow$): $\forall\ \mathfrak{b} \underset{\neq 0}{\triangleleft} A$ invertible $\Rightarrow \mathfrak{b} =$ f.g. $\Rightarrow A =$ noetherian

ONTS: $\Sigma := \{ \mathfrak{b} \underset{\neq}{\triangleleft} A \mid \begin{array}{l} \mathfrak{b} \neq \mathfrak{m}^n \text{ for some } n \\ \mathfrak{b} \neq 0 \end{array} \} = \phi$

Suppose $\Sigma \neq \phi$.

- noetherian $\Rightarrow \exists$ maximal element $\mathfrak{a} \in \Sigma$

$$\Rightarrow \mathfrak{a} \subseteq \mathfrak{m}^{-1}\mathfrak{a} \subseteq \mathfrak{m}^{-1}\mathfrak{m} = A$$

- $\mathfrak{a} = \mathfrak{m}^{-1}\mathfrak{a} \Rightarrow \mathfrak{m}\mathfrak{a} = \mathfrak{m}\mathfrak{m}^{-1}\mathfrak{a} = \mathfrak{a} \Rightarrow \mathfrak{a} = 0$ ↯
- $\mathfrak{a} \neq \mathfrak{m}^{-1}\mathfrak{a} \Rightarrow \mathfrak{m}^{-1}\mathfrak{a} = \mathfrak{m}^n$ for some n

$$\Rightarrow \mathfrak{a} = \mathfrak{m}^{n+1} \quad ↯$$

Thm 9.8 $A =$ integral domain.

$A =$ Dedekind $\Leftrightarrow \forall$ non-zero fractional ideal of A is invertible

Pf: $\Rightarrow$) $\forall\ M =$ fractional

$\Rightarrow M =$ f.g. & $M_{\mathfrak{p}} =$ fractional

$\xRightarrow{A_{\mathfrak{p}} = DVR} M =$ f.g. & $M_{\mathfrak{p}} =$ invertible $\forall\ \mathfrak{p} \neq 0$

$\Rightarrow M =$ invertible

$\Leftarrow$): $\forall\ \mathfrak{b} \overset{\neq 0}{\triangleleft} A$ invertible $\Rightarrow \mathfrak{b} =$ f.g. $\Rightarrow A =$ noetherian

- WNTS: $A_{\mathfrak{q}} = \text{DVR} \quad \forall \mathfrak{q} \neq 0$.

ONTS: $\forall \mathfrak{b} \triangleleft A_{\mathfrak{q}}$ invertible.

$\mathfrak{a} := \mathfrak{b}^c = \mathfrak{b} \cap A \triangleleft A$ (invertible)

$\underset{9.7}{\Rightarrow} \mathfrak{b} = \mathfrak{a}_{\mathfrak{q}}$ invertible.

<u>Cor</u> 9.9. A = Dedekind

$\Rightarrow I := \{\text{non-zero fractional ideals of } A\}$ forms a gp

w.r.t. multiplication

<u>group of ideal of A</u>

9.4 $\Rightarrow$ I = free abelian gp. generated by nonzero primes

$$1 \to U \longrightarrow K^* \longrightarrow I \longrightarrow H \to 1$$

<u>gp of units of A</u> <u>ideal class gp of A</u>

Thm (in # theory) K = # field. $A = \mathcal{O}_K$ Then

i) H = finite &

$\#H = 1 \Leftrightarrow I = P \Leftrightarrow A = \text{PID} \Leftrightarrow A = \text{UFD}$

ii) U = f.g. abelian gp. with rank $r_1 + r_2 - 1$.

Pf: # theory